

Transverse Lattice QCD in 2+1 Dimensions

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Following a suggestion due to Bardeen and Pearson, we formulate an effective light-front Hamiltonian for large- N gauge theory in $(2+1)$ -dimensions. Two space-time dimensions are continuous and the remaining space dimension is discretised on a lattice. Eguchi-Kawai reduction to a $(1+1)$ -dimensional theory takes place. We investigate the string tension and glueball spectrum, comparing with Euclidean Lattice Monte Carlo data.

1. Transverse Lattices

A number of years ago Bardeen and Pearson [1] formulated a light-front Hamiltonian lattice gauge theory, which makes use of the fact that two components of the gauge field are unphysical. In this approach *two* spacetime dimensions are continuous while the remaining ‘transverse’ spatial dimensions are discretised on a lattice. An empirical study [2] of $SU(\infty)$ gauge theory in $(3+1)$ -dimensions produced a rough glueball spectrum, but results were inconclusive. In this exploratory study we bring more recent techniques and ideas to bear on this problem, choosing pure $SU(\infty)$ Yang-Mills theory in $(2+1)$ -dimensions as a trial. We shall unashamedly use as a benchmark the Euclidean Lattice Monte Carlo (ELMC) glueball results of Teper [3]. We have applied analytic and numerical techniques to measure the transverse string tension and glueball spectrum, as well as many other observables not described here.

For $2+1$ dimensions one leaves longitudinal co-ordinates x^0 and x^2 and gauge fields A_0 and A_2 intact, while making the ‘transverse’ co-ordinate x^1 discrete, introducing a Wilson link variable U on each transverse link. Following Ref.[1] we will assume that link variables U below a certain lattice scale have been ‘blocked’ to yield an $N \times N$

complex matrix M_{x^1} on the link between neighboring sites x^1 and $x^1 + a$ of a sub-lattice. This is reminiscent of a Dielectric Lattice Gauge Theory [4], where the magnitude of M plays the role of dielectric constant. In the confining phase, $M = 0$ should be the preferred vacuum solution. The effective potential obtained through blocking is difficult to derive analytically, so we will model it in this work. We choose a Lagrangian density up to 4th order in link fields

$$\begin{aligned} \mathcal{L} = & + \frac{1}{2a^2g^2} \text{Tr} \left\{ D_\alpha M_{x^1} D^\alpha M_{x^1}^\dagger \right\} \\ & - \frac{1}{4g^2} \text{Tr} \{ F_{\alpha\beta} F^{\alpha\beta} \} - V_{x^1}(M) \end{aligned} \quad (1)$$

where

$$\begin{aligned} V_{x^1}(M) = & \mu^2 \text{Tr} \left\{ M_{x^1} M_{x^1}^\dagger \right\} \\ & + \frac{\lambda_1}{aN} \text{Tr} \left\{ M_{x^1} M_{x^1}^\dagger M_{x^1} M_{x^1}^\dagger \right\} \\ & + \frac{\lambda_2}{aN} \text{Tr} \left\{ M_{x^1} M_{x^1}^\dagger M_{x^1-a}^\dagger M_{x^1-a} \right\} \end{aligned} \quad (2)$$

and

$$\begin{aligned} D_\alpha M_{x^1} = & \left[\partial_\alpha + iA_\alpha(x^1) \right] M_{x^1} \\ & - iM_{x^1} A_\alpha(x^1 + a) \end{aligned} \quad (3)$$

$\alpha, \beta \in \{x^0, x^2\}$. \mathcal{L} reduces to the usual $2+1$ Yang Mills density if V is tuned so that $M \rightarrow U/\sqrt{2ag^2}$ as $a \rightarrow 0$. This would seem to require (eventually) $\mu^2 < 0$. Since our quantisation is

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restricted to $\mu^2 > 0$, we should access continuum physics by improving the action/operators. In particular, we implicitly assume (for now) that higher order terms in (1) get progressively smaller (e.g. on dimensional grounds).

In light-front coordinates $x^\pm = (x^0 \pm x^2)/\sqrt{2}$, we take x^+ as canonical time and choose the light-front gauge $A_- = 0$. The theory has a conserved current

$$J_{x^1}^+ = i[M_{x^1} \overset{\leftrightarrow}{\partial}_- M_{x^1}^\dagger + M_{x^1-a}^\dagger \overset{\leftrightarrow}{\partial}_- M_{x^1-a}] \quad (4)$$

at each transverse lattice site x^1 . The field A_+ is constrained, obeying $\partial_-^2 A_+ = g^2 J^+/a$ at each site. Solving this constraint leaves only physical fields, in terms of which the light-front momentum and energy are

$$P^+ = 2 \int dx^- \sum_{x^1} \text{Tr} \left\{ \partial_- M_{x^1} \partial_- M_{x^1}^\dagger \right\}$$

$$P^- = \int dx^- \sum_{x^1} V_{x^1}(M) - \frac{g^2}{2a} \text{Tr} \left\{ J_{x^1}^+ \frac{1}{\partial_-^2} J_{x^1}^+ \right\}$$

There remains residual gauge symmetry under x^- -independent transformations at each site x^1 . The zero mode of the A_+ constraint equation forces the corresponding charge to zero, $\int dx^- J_{x^1}^+ = 0$. This gives a Hilbert space at fixed x^+ formed from all possible closed Wilson loops of link matrices M on the transverse lattice (the x^- co-ordinate of each link field remains arbitrary). Note that at $N = \infty$ we do not distinguish $U(N)$ from $SU(N)$, and the effective gauge coupling is $g^2 N$. Also, since the loop-loop coupling constant is $1/N$, we need not include any more than single loops in the Hilbert space. In fact when $P^1 = 0$ we can simply drop the site indices from M and P^α and the Eguchi-Kawai large- N reduction to a one link transverse lattice becomes apparent [5]. That is, in this frame the theory is isomorphic to one defined on a one-link transverse lattice with periodic boundary conditions, where P^α acts on a basis of zero winding number loops.

It is convenient to work in longitudinal momentum space at $x^+ = 0$

$$M_{ij}(x^-) = \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{dk}{\sqrt{k}} \{ a_{-1,ij}(k) e^{-ikx^-} + (a_{+1,ji}(k))^\dagger e^{ikx^-} \} \quad (5)$$

where the modes satisfy equal- x^+ commutators

$$[a_{\lambda,ij}(k), (a_{\rho,kl}(\tilde{k}))^\dagger] = \delta_{ik} \delta_{jl} \delta_{\lambda\rho} \delta(k - \tilde{k}) \quad (6)$$

$$[a_{l,ij}(k), a_{\rho,kl}(\tilde{k})] = 0 \quad (7)$$

In the last two expressions $i, j \in \{1, \dots, N\}$, $\lambda, \rho \in \{+1, -1\}$, and $(a_{l,ij}(k))^\dagger = (a_l^\dagger(k))_{ji}$. $a_{\pm 1}^\dagger$ creates a link mode with orientation \pm on the lattice. The Eguchi-Kawai reduced states corresponding to $P^1 = 0$ and fixed P^+ can be written as linear combinations of singlet Fock basis states (summation on repeated indices implied)

$$\sum_{\substack{p=|n|, |n|+2, \dots \\ p>0}} \int_0^{P^+} \frac{dk_1 \cdots dk_p}{N^{p/2}} \cdot \delta \left(P^+ - \sum_{m=1}^p k_m \right) f^{\lambda\rho \cdots \sigma}(k_1, \dots, k_p) \cdot \text{Tr} \left\{ a_\lambda^\dagger(k_1) a_\rho^\dagger(k_2) \cdots a_\sigma^\dagger(k_p) \right\} |0\rangle \quad (8)$$

where we set the winding number $n = \lambda + \rho + \cdots + \sigma$ equal to zero. It remains to find the coefficient functions f , cyclically symmetric in their arguments, which diagonalise the Hamiltonian P^- and hence the (mass)² operator $2P^+P^-$ with eigenvalue M^2 .

2. The Boundstate Problem.

The renormalisation of the quantum theory follows that of a 2D gauge theory with adjoint matter, involving only self-energy correction to the propagator through normal ordering of interactions [7]. To diagonalise P^- we employed both the analytic method of using an ansatz for the f 's and the numerical one of discretising the momenta k (DLCQ [8]).

The theory possesses several discrete symmetries. Charge conjugation induces the symmetry $\mathcal{C} : a_{+1,ij}^\dagger \leftrightarrow a_{-1,ji}^\dagger$. There are two orthogonal reflection symmetries \mathcal{P}_1 and \mathcal{P}_2 either of which may be used as 'parity.' If $\mathcal{P}_1 : x^1 \rightarrow -x^1$, we have $\mathcal{P}_1 : a_{+1,ij}^\dagger \leftrightarrow a_{-1,ij}^\dagger$. $\mathcal{P}_2 : x^2 \rightarrow -x^2$, is complicated in light-front formalism. Its explicit operation is known only for free particles, which

we call “Hornbostel parity.” The latter is nevertheless useful since it is often an approximate quantum number and its expectation value can be used to estimate \mathcal{P}_2 [7,9]. Given \mathcal{P}_2 and \mathcal{P}_1 we can determine whether spin \mathcal{J} is even or odd using the relation $(-1)^{\mathcal{J}} = \mathcal{P}_1 \mathcal{P}_2$. If rotational symmetry has been restored in the theory, states of spin $\mathcal{J} \neq 0$ should form degenerate \mathcal{P}_1 doublets $|+\mathcal{J}\rangle \pm |-\mathcal{J}\rangle$ [3]. We use “spectroscopic notation” $|\mathcal{J}|^{P_1 C}$ to classify states.

The parameters μ^2 , λ_1 , and λ_2 of the effective potential $V(M)$ are unknown functions of the dimensionless parameter $ag^2N = 1/\beta$, while g^2N should set the overall mass scale. Empirically they can be fit to a spectrum, then some other quantity of interest predicted (e.g. structure functions), or be fixed by examining Lorentz invariance. To measure the string tension in the x^1 direction, we consider a lattice with n transverse links and periodic boundary conditions. We construct a basis of Polyakov loops or “winding modes” that wind once around this lattice and calculate the lowest eigenvalue M^2 . Because of Eguchi-Kawai reduction, this is equivalent to using Polyakov loops of winding number n on the single-link periodic lattice (Eq. (8) with $n \neq 0$). $\Delta(M)/\Delta(n)$ measures the bare string tension. While this vanishing would signal restoration of translation invariance, it by no means ensures rotational invariance since we have treated the action anisotropically. We can attempt to ensure rotational invariance by forming parity doublets in our glueball spectrum for example.

The behavior of the coefficient functions f in Eq. (8) when any one of the arguments vanishes is

$$\lim_{k_1 \rightarrow 0} f_{\lambda, \rho, \dots, \sigma}(k_1, k_2, \dots, k_p) \propto k_1^s$$

$$\frac{2g^2N}{\pi a} \pi s \tan(\pi s) = \mu^2 \quad (9)$$

Specifying also the number of nodes of f as a function of momenta, one can make a sensible ansatz. To a first approximation (generic $V(M)$) an eigenstate (8) has predominantly a fixed number of link fields p , the mass increasing with p due to the mass term μ^2 in P^- . For a given p , the energy also tends to increase with the number of nodes in the wavefunction f due to the

$\tilde{J}(k)\tilde{J}(-k)/k^2$ term. Thus one expects the lowest two glueball eigenstates to be approximately

$$\int_0^{P^+} dk f_{+1,-1}(k, P^+ - k) \cdot \text{Tr} \left\{ a_{+1}^\dagger(k) a_{-1}^\dagger(P^+ - k) \right\} |0\rangle \quad (10)$$

with the lowest state having positive symmetric $f_{+1,-1}(k, P^+ - k)$, corresponding to 0^{++} , and first excited state having $f_{+1,-1}$ antisymmetric with one zero, corresponding to 0^{--} . The next highest states are either a 4-link state with positive symmetric wavefunctions $f_{+1,+1,-1,-1}$ and $f_{+1,-1,+1,-1}$ or a symmetric 2-link state with $f_{+1,-1}$ having two zeros. In the glueball spectrum we identify the latter states as 0_*^{++} and 2^{++} , although actual eigenstates are a mixture of these.

In our numerical solutions we restrict the number of link fields in our basis (8) to be $p \leq p_{\max}$ and discretise momenta by demanding antiperiodicity of the fields in $x^- \rightarrow x^- + L$. For fixed integer valued cut-off $K = LP^+/(2\pi)$ momenta are labeled by odd half integers $\kappa_m = Kk_m/P^+$, $\sum_m \kappa_m = K$. We diagonalise P^- on a computer and study the system as $p_{\max} \rightarrow \infty$ and $K \rightarrow \infty$. At fixed (p_{\max}, K) we swept the coupling constant space of $V(M)$ and show here some results for the string tension and glueball spectrum.

In general M^2 vs. n plots for winding modes show a good fit to the form $M^2 = An^2 - B$, in agreement with the expectations of string theory. Taking into account all information we have gathered, an acceptable theory occurs only in the “wedge shaped region” $-\lambda_1 \leq \lambda_2 \leq \lambda_1/2$. The renormalised trajectory is most likely to pass through decreasing μ^2 at $\lambda_2 > 0$.

In the glueball spectrum Fig. 2 we label the lowest 2^{--} and second 0^{--} states based on $< p >$. We determine $(-1)^{\mathcal{J}}$ based on Hornbostel parity; the exception is the $|\mathcal{J}|^{+-}$ sector where Hornbostel parity gave exactly the opposite of the desired (i.e. Teper’s) results. Although Fig. 2 indicates that qualitative agreement can be obtained with the ELMC data, there is alarming discrepancy from the expected degenerate parity doublet $2^{\pm+}$. This discrepancy is responsible for almost all of the χ^2 error in our fitting procedure. There are undoubtedly errors associ-

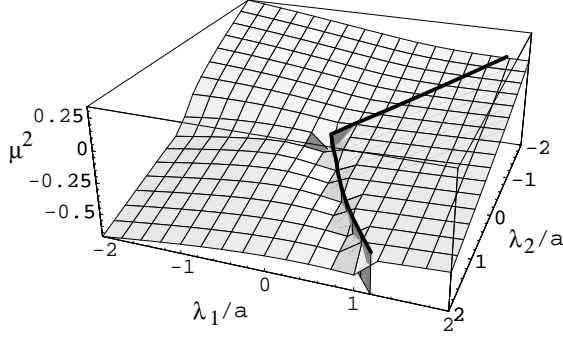


Figure 1. Parameters such that the lowest M^2 eigenvalues are equal for $n = 4$ and 5 winding modes, where $p_{\max} = n + 4$, and $K = 10.5$ or 11. This is an estimate of vanishing bare string tension. Also shown is a line such that the M^2 eigenvalues are approximately degenerate for $n = 3, 4$, and 5.

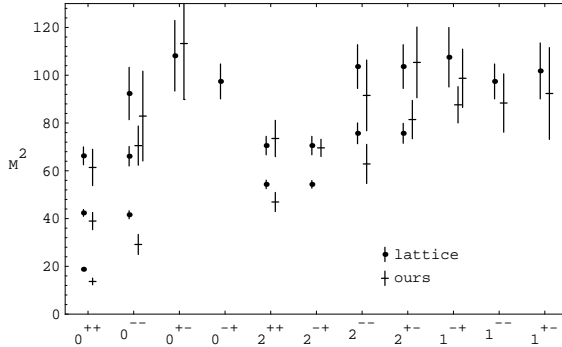


Figure 2. A comparison of our low-lying spectrum with $SU(3)$ ELMC data in units of the physical string tension [3] for various $|J|^{P_1 C}$. The parameters $g^2 N/a = 3.44$, $a\mu^2 = 0.2g^2 N$, $\lambda_1 = 0.34g^2 N$, and $\lambda_2 = 1.27g^2 N$ were chosen by a best fit to the lattice data, $\chi^2 = 45$, where $p_{\max} = 6$ and $K = 14$. Our error estimates are solely for the purpose of performing the χ^2 fit.

ated with K and p_{\max} , but we do not feel that they are sufficient to account for the differences. In fact, we have examined spectra for $K = 10$ and $p_{\max} = 4, 6, 8$, extrapolating to large p_{\max} , along with $p_{\max} = 6$ and $K = 10, 11, 12, 13, 14$, extrapolating to large K . Comparing with large N extrapolated ELMC spectra, in either case we saw no great improvement in our results. There is another quartic term we could have added, $\frac{1}{N^2}(\text{Tr } M^\dagger M)^2$, which gives non-zero contribution only on the link—anti-link Fock state. It improves the parity degeneracies at the expense of a less good ‘radial’ excitation spectrum.

Clearly it is necessary to check the effect of higher order terms in the effective potential $V(M)$ to see if they are small and capable of accounting for the discrepancies between our results and Teper’s. If agreement can be obtained in the clean environment of pure glue in $2 + 1$ dimensions, we see no reason why the same methods cannot be applied in practice to spectra, form factors, and structure functions in $3 + 1$ dimensions.

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REFERENCES

1. W. A. Bardeen and R. B. Pearson, Phys. Rev. D **14** 547 (1976).
2. W. A. Bardeen, R. B. Pearson, and E. Rabinovici, Phys. Rev. D **21** 1037 (1980).
3. M. Teper, Phys. Lett. **289B** 115 (1992); Phys. Lett. **311B** 223 (1993); these proceedings.
4. G.Mack, Nucl. Phys. 235 **B** (1984) 197.
5. S. Dalley and T. R. Morris, Int. Journal Mod. Phys. **A5** 3929 (1990).
6. T. Eguchi and H. Kawai, Phys. Rev. Lett. **48** 1063 (1983).
7. F. Antonuccio and S. Dalley, Nucl. Phys. B **461** 275 (1996).
8. H.-C. Pauli and S. Brodsky, Phys. Rev. D **32** 1993 and 2001 (1985).
9. B. van de Sande and M. Burkardt, Phys. Rev. D **53** 4628 (1996).